Problem A.22

Consider the matrix

$$\mathsf{M} = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}.$$

(a) Is it normal?

(b) Is it diagonalizable?

Solution

Calculate the hermitian conjugate of M.

$$\mathsf{M}^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix}$$

Since

$$\mathsf{M}\mathsf{M}^\dagger = \begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix} \neq \begin{pmatrix} 2 & 1+i \\ 1-i & 2 \end{pmatrix} = \mathsf{M}^\dagger\mathsf{M},$$

the matrix M is not normal. Consider the eigenvalue problem for M.

$$\mathsf{M}\mathsf{a} = \lambda\mathsf{a}$$

Bring $\lambda\mathsf{a}$ to the left side and combine the terms.

$$(\mathsf{M} - \lambda\mathsf{I})\mathsf{a} = \mathsf{0}$$

Since $\mathsf{a} \neq \mathsf{0}$, the matrix in parentheses must be singular, that is,

$$\det(\mathsf{M} - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & i-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(i-\lambda) - 1 = 0$$

$$\lambda^2 - (1+i)\lambda + (i-1) = 0$$

$$\lambda = \frac{(1+i) \pm \sqrt{(1+i)^2 - 4(i-1)}}{2} = \frac{(1+i) \pm \sqrt{4-2i}}{2}.$$

Since there are two distinct eigenvalues, there are two corresponding eigenvectors, meaning the 2×2 matrix M is normalizable.